SYDNEY GRAMMAR SCHOOL

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

MATHEMATICS 3 UNIT

Time allowed: 2 hours. 16th August 1991

All questions may be attempted.
All questions are of equal value.
All necessary working should be shown.
Marks may not be awarded for careless or badly arranged work.
Approved calculators may be used.
Write your examination number on each sheet of paper.
Hand in each question separately.
Hand in a cover sheet for each question, even any not attempted.

The following Standard Integrals may be used.

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1},\ n \neq -1;\ x \neq 0,\ \text{if } n < 0.$$

$$\int \frac{1}{x}\,dx = \log_e x,\ x > 0.$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax},\ a \neq 0.$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax,\ a \neq 0.$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax,\ a \neq 0.$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax,\ a \neq 0.$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax,\ a \neq 0.$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\ a \neq 0.$$

$$\int \frac{1}{\sqrt{(a^2-x^2)}}\,dx = \sin^{-1}\frac{x}{a},\ a > 0,\ -a < x < a.$$

$$\int \frac{1}{\sqrt{(x^2-a^2)}}\,dx = \log_e\{x+\sqrt{(x^2-a^2)}\},\ |x| > |a|.$$

$$\int \frac{1}{\sqrt{(x^2+a^2)}}\,dx = \log_e\{x+\sqrt{(x^2+a^2)}\}.$$

QUESTION 1.

(a) Factorize $a^3 - 8$.

(b) Solve for x: $\frac{1}{x} > 4$.

(c) Simplify and write with a rational denominator:

$$\frac{1}{(\sqrt{3}-2)^2}.$$

(d) If A and B are the points $(-1,4)$ and $(3,-3)$ respectively, find the co-ordinates of the point P which divides AB <u>externally</u> in the ratio 4:1.

(e) (i) Differentiate $\cos^{-1}(4x)$.

(ii) If $y = \sin 2x$, show that $\frac{d^2y}{dx^2} = -4y$.

QUESTION 2.

(a) Use the substitution $u = x^2 - 1$ to evaluate $\int_1^2 2x(x^2-1)^4\,dx$.

(b) (i) Use the change of base formula to express $\log_2 x$ in terms of $\log_e x$.

(ii) Hence write down the derivative of $\log_2 x$.

(c) (i) Express sinA and cosA in terms of 't', where t is $\tan\frac{1}{2}A$.

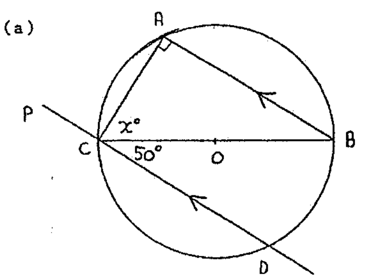
(ii) Hence or otherwise prove that $\frac{1+\cos 2A}{\sin 2A} = \cot A$.

(d) Use mathematical induction to prove that:

$$8 + 16 + 26 + 40 + \ldots + (2^n + 6n) = 2^{n+1} + 3n(n+1) - 2.$$

QUESTION 3.

(a)



In the diagram, O is the centre of the circle, CB is a diameter and PD∥AB. ∠BCD = 50°, ∠ACB = x°.

(i) Copy the diagram into your writing booklet.
(ii) Find x, giving all reasons.

continued over/.........

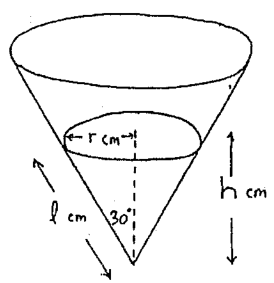
Question 3 continued.

(b) The equation of a circle is given by
$x^2 + y^2 + 6x - 2y + 6 = 0$.

(i) By expressing the equation in the form $(x-h)^2 + (y-k)^2 = a^2$, find the centre and radius of the circle.

(ii) Show algebraically that the line $y = 3$ is a tangent to the circle.

(c)



The diagram shows a conical vessel with a semi-vertical angle of 30° which is being filled with water at the rate of 20 cm³/min. When the amount of water in the vessel is V cm³ the radius of the water surface is r cm, the perpendicular height is h cm and the slant height is ℓ cm, as shown in the diagram.

(i) Show that $V = \frac{\pi \ell^3 \sqrt{3}}{24}$

(For a cone, $V = \frac{1}{3}\pi r^2 h$).

(ii) Find the rate at which the slant height ℓ is increasing when the radius of the water surface is 6cm.

QUESTION 4.

(a) Without calculus sketch the graph of the polynomial $P(x) = (x+2)^2(4-x)(x-1)$. Hence or otherwise solve the inequality $(x+2)^2(4-x)(x-1) \leqslant 0$.

(b) The acceleration of a particle P moving in a straight line is given by $\ddot{x} = -3\sqrt{x}$ m/s² where x metres is the displacement of P from the origin O. If P is at rest when 9m to the right of O, find the ~~velocity~~ speed of P when it is 4m to the right of O.

continued over/.......

Question 4 continued.

(c) The population N of an ant colony at time t days after time $t = 0$ is given by $N = 2.4 \times 10^7 + Ae^{0.03t}$

(i) Show by differentiation that $\frac{dN}{dt} = k(N - B)$ where k, B are constants.

(ii) Initially the ant population was 6.7×10^7. Find the population (2 sig. figs.) after 10 days.

(iii) Find (nearest day) the time taken for the population to treble.

QUESTION 5.

PQ is a chord on the parabola $x^2 = 4ay$ where P,Q are the points $(2ap, ap^2)$, $(2aq, aq^2)$ respectively.

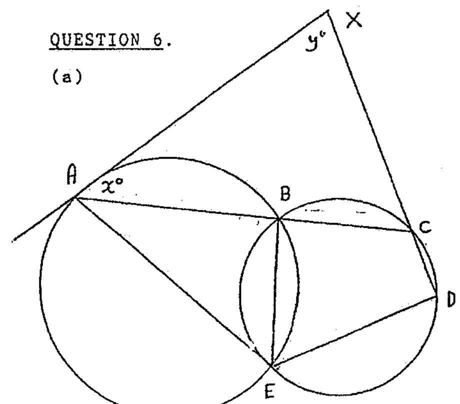
(i) Derive the equation of the tangent at P. Hence write down the equation of the tangent at Q. If the tangents intersect at the point T, show that the co-ordinates of T are $(a(p+q), apq)$.

(ii) If PQ is a focal chord, show that $pq = -1$ and hence prove that T lies on the directrix of the parabola and that $\angle PTQ = 90^\circ$.

*(iii) If S is the focus of the parabola, find the co-ordinates of M the mid point of QS. Show that the locus of the point M is a parabola and find its vertex.

QUESTION 6.

(a)



Two circles intersect at B, E and AX is a tangent. AB meets the second circle at C, and XC meets the second circle again at D as shown.

$\angle XAC = x^\circ$ and $\angle AXC = y^\circ$.

(i) Copy the diagram into your writing booklet.
(ii) Give the reason why $\angle AEB = x$.
(iii) Prove that AXDE is a cyclic quadrilateral.

continued over/.......

Question 6 continued

(b) (i) Use the binomial theorem to write out the expansion of $(1+x)^n$.

(ii) Hence prove that $\sum_{r=1}^{n} r\ {}^{n}C_{r}\, 2^{r-1} = n.3^{n-1}$

(c) Prove that $\tan^{-1}x + \cot^{-1}(x+1) = \tan^{-1}(x^2+x+1), \quad x \geqslant 0$.

R.N.

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Excellent work!

1.(a). $a^3 - 8$
$= (a-2)(a^2 + ab + 4)$ ✓

(b). $\frac{1}{x} > 4$ $x^2 \times \frac{1}{x} > 4 \times x^2$
$x < \frac{1}{4}$ ½ of the answer $x > 4x^2$ ... try again.

(d). $A(-1, 4)$ $B(3, -3)$ 4:1
$\left(\frac{12+1}{5}, \frac{-12-4}{5}\right)$
$= \left(\frac{13}{5}, -\frac{16}{5}\right)$ ✓

(c). $\frac{1}{(\sqrt{3}-2)^2} = \frac{1}{3 - 2\sqrt{3} + 4}$
$= \frac{1}{7 - 2\sqrt{3}} \times \frac{7 + 2\sqrt{3}}{7 + 2\sqrt{3}}$ ✓
$= \frac{7 + 2\sqrt{3}}{37}$ ✓

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(e) (i) $\cos^{-1} 4x$
$= \frac{-4}{\sqrt{1 - 16x^2}}$ ✓

(ii). $y = \sin 2x$
$y' = 2\cos 2x$ ✓
$y'' = -4\sin 2x$ ✓
$= -4y$

2(a). $\int_1^2 2x(x^2-1)^4 \, dx$ $= \left[\frac{(x^2-1)^5}{5}\right]_1^2$
Let $u = x^2 - 1$ At $x = 2, u = 3$ $= \frac{243}{5}$
$du = 2x \, dx$ ✓ $x = 1, u = 0$.
$I = \int_0^3 u^4 \, du$
$= \left[\frac{u^5}{5}\right]_0^3$
$= 243/5$ ✓

(b)(i). $\log_2 x = \frac{\ln x}{\ln 2}$ ✓

(ii) $\frac{1}{x \ln 2}$ ✓

(c) (i) $\sin A$
Let $t = \tan \frac{A}{2}$
$\sin 2A/2 = 2\cos \frac{A}{2} \sin \frac{A}{2}$
$= \frac{2t}{1+t^2}$ ✓

$\cos A$
$= \cos^2 \frac{A}{2} - \sin^2 \frac{A}{2}$
$= \frac{1-t^2}{1+t^2}$ ✓

(ii). $\frac{1 + \cos 2A}{\sin 2A}$
$= \frac{1 + 2\cos^2 A - 1}{2\sin A \cos A}$ ✓
$= \frac{2\cos A \cos A}{2\sin A \cos A}$ ✓
$= \cot A$

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(d). $8 + 16 + 26 + \ldots (2^n + 6n) = 2^{n+1} + 3n(n+1) - 2$
Let $n = 1$
LHS: 8   RHS: $2^2 + 3(2) - 2$
$= 8$.
True ✓

Let $n = k$
$8 + 16 + \ldots (2^k + 6k) = 2^{k+1} + 3k(k+1) - 2$
Let $n = k+1$
$8 + 16 + \ldots (2^k + 6k) + (2^{k+1} + 6(k+1)) = 2^{k+2} + 3(k+1)(k+2) - 2$
LHS $= 2^{k+1} + 3k(k+1) - 2 + 2^{k+1} + 6(k+1)$ ✓
$= 2 \cdot 2^{k+1} + 3k^2 + 3k + 6k + 4$
$= 2^{k+2} + 3k^2 + 9k + 4$
$= 2^{k+2} + 3k^2 + 9k + 6 - 2$ ✓
$= 2^{k+2} + 3(k+1)(k+2) - 2$
= RHS

∴ If true for n=k, then shown for n=k+1, by the principle of Mathematical Induction, true for all integers.

3. (a). $A\hat{C}D = \pi/2$ (co-interior Angles) ✓
∴ $A\hat{C}B = x = 40°$ (Angle sum).

(b)(i) $x^2+y^2+6x-2y+6=0$.

$(x^2+6x+9)+(y^2-2y+1)=-6+9+1$

$(x+3)^2+(y-1)^2=4$

Centre $(-3,1)$ radius 2.

(ii) Sub $y=3$

$x^2+9+6x-6+6=0$.

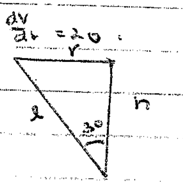
$x^2+6x+9=0$.

$\Delta = 36-4\times 9$
$=0$.

∴ Tangent.

(12)

(c). $\frac{dV}{dt}=20$.



$\tan 30 = r/h$

$\frac{1}{\sqrt{3}} = r/h$

$h = r\sqrt{3}$

$V = \frac{\pi r^2 h}{3}$

$= \frac{\pi r^3\sqrt{3}}{3}$

$= \frac{\pi r^3}{\sqrt{3}}$

$= \frac{\pi l^3}{8\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$

$= \frac{\pi l^3\sqrt{3}}{24}$.

$\sin 30 = \frac{r}{l}$

$\frac{1}{2} = \frac{r}{l}$

$2r = l$

$r = l/2$.

(ii). $\frac{dV}{dl} = \frac{3\pi l^2\sqrt{3}}{24}$

$\frac{dl}{dt} = \frac{dl}{dV} \times \frac{dV}{dt}$

$= \frac{24}{3\sqrt{3}\pi l^2} \times 20$.

$= \frac{160}{\sqrt{3}\pi l^2}$

when $r=6$

$\sin 30 = 6/l$

$l = 12$.

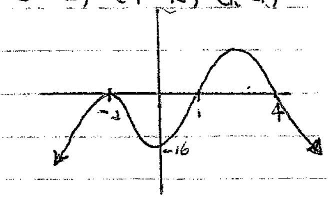
At $l=12$

$= \frac{160}{144\pi\sqrt{3}}$

$= \frac{10}{9\pi\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$

$= \frac{10\sqrt{3}}{27\pi}$

(a) $P(x) = (x+2)^2(4-x)(x-1)$



$P(x) \leq 0$.

$x \leq -1 \cap x \geq 4$.

(b) $\ddot{x} = -3\sqrt{x}$

$\frac{d}{dx}(\frac{1}{2}v^2) = -3\sqrt{x}$

$\frac{1}{2}v^2 = \frac{-3x^{3/2}}{3/2}$

$\frac{1}{2}v^2 = -2x^{3/2} + c$.

$v^2 = -4x^{3/2} + c$.

At $x=9, v=0$.

$0 = -108 + c$

∴ $c = 108$.

$v^2 = -4x^{3/2} + 108$

At $x=4$,

$v^2 = -32 + 108$

$v^2 = 76$

$v = 2\sqrt{19}$

(11)

(c). (i) $N = 2.4\times10^7 + Ae^{0.003t}$ $k = 0.003$, $B = 2.4\times10^7$

$\frac{dN}{dt} = 0.003\,Ae^{0.003t}$ $N - B = Ae^{kt}$

$= k(N-B)$

(ii) $t=0$, $N = 6.7\times10^7$

$6.7\times10^7 = 2.4\times10^7 + A$

$A = \frac{67}{24}$.

At $t=10$,

$N = 2.4\times10^7 + \frac{67}{24}e^{0.03}$

$= 24\ 000\ 000$ (2 sig figs)

$= 2.4\times10^7$

(iii) For $N = 7.2\times10^7$

$7.2\times10^7 = 2.4\times10^7 + \frac{67}{24}e^{0.003t}$

$24 \times \frac{4.8\times10^7}{67} = e^{0.003t}$

$16.66 = 0.003t$ ∴ $t = 555.3$

5.(a). (i) $x^2 = 4ay$

$y = \frac{x^2}{4a}$

$\frac{dy}{dx} = \frac{2x}{4a}$

At $x = 2ap$ ✓

$m = p$.

Tag: $y - ap^2 = p(x - 2ap)$

$y - ap^2 = px - 2ap^2$

$y = px - ap^2$

At Q: $y = qx - aq^2$

∴ T: $px - ap^2 = qx - aq^2$

$aq^2 - ap^2 = qx - px$

$a(q-p)(q+p) = x(q-p)$

$x = a(p+q)$

Sub into (P)

$y = pa(p+q) - ap^2$

$= ap^2 + apq - ap^2$

$= apq$ ✓

∴ T $(a(q+p), apq)$

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(ii) $m_{PQ} = \frac{aq^2 - ap^2}{2aq - 2ap}$

$= \frac{a(q-p)(q+p)}{2a(q-p)}$

$= \frac{p+q}{2}$

$y - ap^2 = \left(\frac{p+q}{2}\right)x + (p+q)ap$

Focal chord ie (0, a)

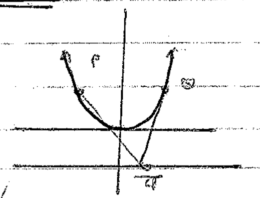
$a - ap^2 = ap^2 + apq$

$a = -apq$ ✓

$pq = -1$

∴ T = $(a(p+q), -a)$

$y = -a$, ∴ Directrix. ✓



∴ T is intersection of P & Q

ie gradients $= pq = -1$ ✓

∴ Intersect at $\pi/2$.

(iii) $S(0, a)$ $Q(2aq, aq^2)$

$x = aq$, $y = \frac{a(q^2-1)}{2}$

$x/a = q$

$y = \frac{a((x/a)^2 - 1)}{2}$ ✓

$2y = \frac{ax^2}{a} - a$

~~$2ay = ax^2 - a^2$~~

~~$2ay = a(x^2 - a)$~~

$2ay = x^2 - a^2$ — Parabola.

~~$x^2 = 2ay + a^2$~~

~~$= 4a\left(\frac{1}{2}y + \frac{a}{4}\right)$~~

~~∴ Vertex $\frac{1}{2}y + \frac{a}{4} = 0$~~

~~$\frac{y^2}{2} + \frac{a}{2} = 0$~~

~~$y =$~~

$y = \frac{x^2}{2a} - \frac{a}{2}$.

$\frac{dy}{dx} = \frac{2x}{2a}$

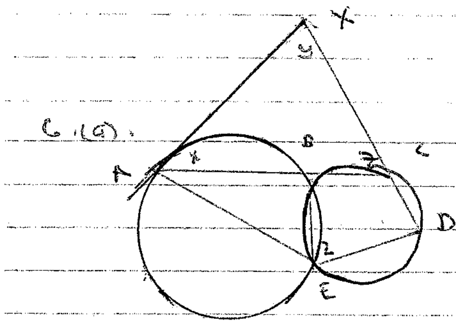
$= \frac{x}{a} = 0$. ✓

$x = 0$.

∴ $y = -a/2$ ✓

Vertex ∴ $(0, -a/2)$ ✓

6.(a).



(ii) $A\hat{E}B = x$ (Alternate Segment) ~~Angles subtended in the~~

(iii). Let $A\hat{C}X = z$

∴ $D\hat{E}D = z$ (exterior cyclic quad)

$A\hat{E}B = x$ (prove above)

∴ $A\hat{E}D = x + z$ (Angle)

From ΔAXC, $x + z + y = \pi$ ✓

$A\hat{X}C + A\hat{E}D = x + z + y = \pi$

∴ AXDE is concyclic ✓

(Opposite Angles add to π)

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(b).(i) $(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \dots \binom{n}{r}x^r + \binom{n}{n}x^n$

(ii) $\frac{d}{dx} = n(1+x)^{n-1} = \binom{n}{1} + 2\binom{n}{2}x + 3\binom{n}{3}x^2 + \dots r\binom{n}{r}x^{r-1} + \dots n\binom{n}{n}x^{n-1}$

Let $x = 2$. ✓

LHS $= n3^{n-1}$

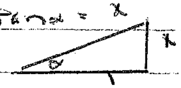
RHS $= \binom{n}{1} + 2\times2\binom{n}{2} + 2^2 3\binom{n}{3} + 2^3\times4\binom{n}{4} + \dots n2^{n-1}\binom{n}{n}$

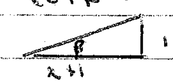
$= \sum_{r=1}^{n} r\binom{n}{r}2^{r-1}$ ✓

(c). $\tan^{-1}x + \cot^{-1}(x+1) = \tan^{-1}(x^2+x+1)$

Let $\tan^{-1}x = \alpha$ $\cot^{-1}(x+1) = \beta$

$\tan\alpha = x$ $\cot\beta = x+1$





$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$ ✓

$= \frac{x + \frac{1}{x+1}}{1 - \frac{x}{x+1}} = \frac{x^2+x+1}{x+1-x}$

$= x^2 + x - 1$

∴ $\alpha + \beta = \tan^{-1}(x^2+x+1)$ ✓